Eur. Phys. J. C 4, 693–704 (1998)
DOI 10.1007/s100529800803

THE EUROPEAN PHYSICAL JOURNAL C

Broad sub-continuum resonances and the case for finite-energy sum-rules

A.S. Deakin[1,a], V. Elias[1,b], A.H. Fariborz[1,c], Ying Xue[1,d], Fang Shi[2,e], T.G. Steele[2,f]

[1] Department of Applied Mathematics, The University of Western Ontario, London, Ontario N6A 5B7, Canada (e-mail: [a] asd@apmaths.uwo.ca, [b] zohar@apmaths.uwo.ca, [c] amir@pineapple.apmaths.uwo.ca, [d] xue@pineapple.apmaths.uwo.ca)

[2] Department of Physics and Engineering Physics, University of Saskatchewan, Saskatoon, Saskatchewan S7N 5C6, Canada (e-mail: [e] sfang@skatter.usask.ca, [f] steelet@sask.usask.ca)

Received: 15 September 1997 / Revised version: 15 November 1997 / Published online: 26 February 1998

Abstract. There is a need to go beyond the narrow resonance approximation for QCD sum-rule channels which are likely to exhibit sensitivity to broad resonance structures. We discuss how the first two Laplace sum rules are altered when one goes beyond the narrow resonance approximation to include possible subcontinuum resonances with nonzero widths. We show that the corresponding first two finite energy sum rules are insensitive to the widths of such resonances, provided their peaks are symmetric and entirely below the continuum threshold. We also discuss the reduced sensitivity of the first two finite energy sum rules to higher dimensional condensates, and show these sum rules to be insensitive to dimension > 6 condensates containing at least one $\bar{q}q$ pair. We extract the direct single-instanton contribution to the F_1 sum rule for the longitudinal component of the axial-vector correlation function from the known single-instanton contribution to the lowest Laplace sum rule for the pseudoscalar channel. Finally, we demonstrate how inclusion of this instanton contribution to the finite-energy sum rule leads to both a lighter quark mass and to more phenomenologically reasonable higher-mass-resonance contributions within the pseudoscalar channel.

1 Introduction: Nonzero resonance widths and QCD Laplace sum-rules

Hadron properties can be extracted by relating phenomenological and field-theoretical expressions for integrals over appropriately chosen current-correlation functions, integrals which we denote as QCD sum rules [1]. In the narrow resonance approximation, hadronic contributions to the imaginary part of current-current correlation functions are proportional to δ-functions at the resonance mass,

$$Im[\Pi^h(s)] = \sum_r \pi g_r \delta(s-m_r^2) + \Theta(s-s_0) Im[\Pi^p(s)], \quad (1)$$

The summation in (1) is over all resonances r in the channel under consideration such that m_r^2 is less than s_0. Above this hadron-continuum threshold, the hadronic contribution $\Pi^h(s)$ to the correlation function is assumed to be the same as the contribution $\Pi^p(s)$ from perturbative QCD, as is evident from (1).

The hadronic sub-continuum (h) contribution to the kth Laplace sum rule, corresponding to the transform of the appropriate portions of (1), is defined to be

$$R_k^h(\tau) \equiv \int_0^\infty ds (1/\pi) Im[\Pi^h(s) - \Pi^p(s)\Theta(s-s_0)] s^k \, e^{-s\tau} \quad (2)$$

In the narrow resonance approximation ($\Gamma \to 0$), we see from (1) that

$$\lim_{\Gamma \to 0} R_k^h(\tau) = \sum_r g_r m_r^{2k} \exp[-m_r^2 \tau], \quad (3)$$

an expression in which contributions from more-massive resonances are exponentially suppressed. Note from (3) that $R_1^h(\tau) \geq m_\ell^2 R_0^h(\tau)$ where m_ℓ denotes the mass of the lowest-lying resonance in the channel. Consequently, $R_1^h(\tau)/R_0^h(\tau)$ is bounded from below by m_ℓ^2. Standard QCD sum-rule methodology involves minimizing this ratio [or its field-theoretical analogue] with respect to τ in order to determine a value of m_ℓ^2 [2]. The sum rule $R_k^h(\tau)$ corresponds to the following field-theoretical contribution from perturbative-QCD and nonperturbative (np) QCD-vacuum effects:

$$R_k^{QCD}(\tau) = \int_0^{s_0} ds (1/\pi) Im[\Pi^p(s)] s^k e^{-s\tau} + (-\partial/\partial\tau)^k \left\{ (1/\tau) \mathcal{L}_\tau^{-1} [-d\Pi^{np}(s)/dQ^2] \right\}. \quad (4)$$

In (4), $Q^2 \equiv -s$, and $\Pi^{np}(s)$ represents all correlation-function contributions from QCD-vacuum condensates as well as direct instanton contributions. The inverse Laplace transform in (4), corresponding to the Laplace-transform definition

$$\mathcal{L}_{Q^2}[f(\tau)] \equiv \int_0^\infty d\tau f(\tau) e^{-Q^2\tau}, \quad (5)$$

is utilized to take advantage of the operator-product expansion of Π^{np} in inverse powers of Q^2, and is easily understood via dispersion-relation methodology:

$$(1/\tau)\mathcal{L}_\tau^{-1}[-d\Pi^{np}(s)/dQ^2] \quad (6)$$
$$= (1/\tau)\mathcal{L}_\tau^{-1}[(1/\pi)\int_0^\infty ds Im[\Pi^{np}(s)]/(s+Q^2)^2]$$

As is evident from (5), $\mathcal{L}_\tau^{-1}[1/(s+Q^2)^2] = \tau e^{-s\tau}$, which, upon substitution into (6) and (4), leads to a result consistent with duality between QCD $[\Pi^p(s) + \Pi^{np}(s)]$ and phenomenological hadronic physics $[\Pi^h(s)]$:

$$R_k^{QCD}(\tau) + \int_{s_0}^\infty ds(1/\pi)Im[\Pi^p(s)]s^k e^{-s\tau}$$
$$= \int_0^\infty ds(1/\pi)Im[\Pi^p(s) + \Pi^{np}(s)]s^k e^{-s\tau} \quad (7)$$

Duality between $R_k^{QCD}(\tau)$ and $R_k^h(\tau)$ then follows via comparison of (7) and (2). The mass of the lowest lying resonance can be determined via the relationship

$$Min[R_1^{QCD}(\tau)/R_0^{QCD}(\tau)] \geq m_\ell^2 \quad (8)$$

over an appropriate range of $\tau[s_0^{1/2} > \tau^{-1/2} >> \Lambda_{QCD}]$.

There is a need to go beyond the narrow resonance approximation if QCD sum rules exhibit sensitivity to resonance structures with non-zero widths. Such structures can not always be absorbed in the sum-rule continuum – even the lowest hadronic resonances may have substantial widths. For example, theoretical arguments exist [3,4] for the first pion-excitation to have a mass below 1 GeV, a floor for any reasonable estimate of the continuum threshold above which perturbative and hadronic QCD should coincide. Even if the first pion excitation state is identified with the $\Pi(1300)$ resonance, whose mass pole is still likely to be below the continuum threshold, the width of this resonance may be as large as 600 MeV[5].

To gain qualitative insight into how nonzero resonance widths can effect QCD sum rule calculations, we can replace the δ-function within resonance contributions to (1) with a rectangular pulse of unit area centred at $s = m^2$ with full-width $\Delta s = 2m\Gamma$:

$$\delta(s-m^2) \to \mathcal{P}_m(s,\Gamma) \quad (9)$$
$$\equiv [\Theta(s-m^2+m\Gamma) - \Theta(s-m^2-m\Gamma)]/2m\Gamma.$$

Let us consider how such an approximation to a lowest-lying resonance alters a QCD Laplace sum-rule determination of that resonance's mass. We assume that all but the lowest-lying (ℓ) resonance is absorbed in the continuum. If we replace the delta function for the lowest-lying resonance with the pulse $\mathcal{P}_m(s,\Gamma)$, we find from (2) that

$$R_k^h(\tau) = g_\ell \int_0^{s_0} ds \mathcal{P}_m(s,\Gamma) s^k e^{-s\tau}$$
$$= g_\ell m^{2k} e^{-m^2\tau} \Delta_k(m,\Gamma,\tau), \quad (10)$$

with the functions $\Delta_{0,1}$ found from explicit evaluation of the integrals in (10)

$$\Delta_0(m,\Gamma,\tau) = \sinh(m\Gamma\tau)/(m\Gamma\tau),$$
$$\Delta_1(m,\Gamma,\tau) = \Delta_0(m,\Gamma,\tau)[1 + 1/(m^2\tau)]$$
$$- \cosh(m\Gamma\tau)/m^2\tau). \quad (11)$$

Note that $\Delta_{0,1}(m,\Gamma,\tau) \to 1$ as $\Gamma \to 0$, consistent with the δ-function limit (9). We see immediately from (10) and (11) that

$$m^2 = \left[R_1^h(\tau)/R_0^h(\tau)\right]\left[\Delta_0(m,\Gamma,\tau)/\Delta_1(m,\Gamma,\tau)\right]$$
$$= [m^2]_{\Gamma=0} \cdot [1 + \Gamma^2\tau/3 + O(\Gamma^4)]. \quad (12)$$

Since $R_1^h(\tau)/R_0^h(\tau)$ corresponds to R_1^{QCD}/R_0^{QCD} by duality, this latter ratio corresponds to m^2 in the narrow resonance approximation ($\Gamma = 0$). We see from (12) that finite width effects will *increase* the masses of lowest-lying resonances extracted via Laplace sum rules.

For Laplace sum rules, a more quantitative estimate of resonance-width effects could be obtained by replacing the delta-functions in (1) with Breit-Wigner peaks, and then substituting into the Laplace sum-rule definition (2). However, the Breit-Wigner shape has an infinite tail, and significant portions of that tail may extend above the continuum threshold s_0 or below the $s = 0$ boundary into Euclidean momenta. Such contributions from the Breit-Wigner tail, whether included *or* truncated away, can be genuinely substantial for resonances with widths in excess of 100 MeV, and can be a source of *theoretical uncertainty* in Laplace sum-rule analyses of broad sub-continuum resonances. Such uncertainty may be understood as a limitation on Laplace sum-rule methodology itself, particularly for channels in which more than one resonance lies below the continuum threshold. Non-lowest-lying resonances are expected to be less stable, and consequently, to be substantially broader than lowest-lying resonances. The I = 1 pseudoscalar channel has already been mentioned as an example of such a channel, and is discussed in the final two sections of this paper.

2 Nonzero resonance widths and finite-energy sum-rules

For a given current-correlation function $\Pi(s)$, the finite-energy sum-rules (FESR's) $F_k(s_0)$ are defined here to be the integrals [6]

$$F_k(s_0) \equiv (1/2\pi i)\int_{C(s_0)} ds\, s^k \Pi(s)$$
$$= (1/\pi)\int_0^{s_0} ds\, s^k Im[\Pi(s)], \quad (13)$$

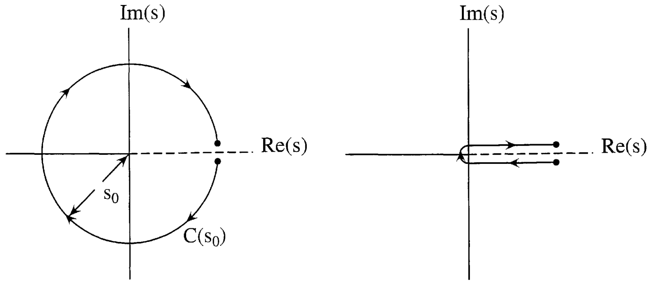

Fig. 1. a The contour $C(s_0)$, **b** distortion of $C(s_0)$ to enclose the positive real s-axis

where the contour $C(s_0)$ is an open circle of radius s_0 in the complex s-plane that does not cross the real s-axis [Fig. 1a]. The parameter s_0 is understood to be the continuum threshold discussed in the previous section. As indicated in (13), the contour $C(s_0)$ can be distorted into a line running below and above the physical singularities on the positive real s-axis [Fig. 1b].

In the narrow resonance approximation (1), one finds that

$$F_k^h(s_0) = \sum_r g_r m_r^{2k} \equiv \sum_r [F_k^h(s_0)]_r \qquad (14)$$

an expression that differs from (3) only in that higher-mass sub-continuum resonances are no longer exponentially suppressed. This is a positive feature of the FESR approach, if one is seeking to use sum rules to obtain information about such resonances.

To examine finite width effects, let us first replace the delta-functions of (1) with the finite-width rectangular pulses (9). As long as $s_0 \geq m_r^2 + 2m_r\Gamma_r$, the contribution of such a pulse to F_0 is clearly the same as that of a delta-function, since F_0 is sensitive only to peak-area:

$$[F_0^h(s_0)]_r \to \int_0^{s_0} ds\ g_r \mathcal{P}_{m_r}(s, \Gamma_r) = g_r. \qquad (15)$$

Remarkably, the F_1 sum-rule is *also* insensitive to the width of the rectangular pulse:

$$[F_1^h(s_0)]_r \to \int_0^{s_0} ds\ s\ g_r \mathcal{P}_{m_r}(s, \Gamma_r)$$
$$= [g_r/(2m_r\Gamma_r)] \int_{m_r^2 - m_r\Gamma_r}^{m_r^2 + m_r\Gamma_r} ds\ s = g_r m_r^2 \qquad (16)$$

The final result of (16) is identical to the contribution to $[F_1^h(s_0)]_r$ obtained from the narrow resonance approximation, with $\mathcal{P}_{m_r}(s, \Gamma_r)$ replaced by $\delta(s - m_r^2)$. The results (15) and (16) are to be contrasted with the width-dependence exhibited in (10) and (11) for corresponding Laplace sum rules.

Moreover, the width-independence of the first two FESR's obtained above is *not* an artifact of the rectangular pulse approximation for non-zero width resonances. Any symmetric resonance peak $\mathcal{R}_m(s)$ centred at m^2 can be represented as a sum over variable-width unit-area rectangular pulses $\mathcal{P}_m(s, \Gamma')$ centred at $s = m^2$:

$$\mathcal{R}_m(s) = \int_0^{\Gamma_{max}} d\Gamma' f(\Gamma') \mathcal{P}_m(s, \Gamma') \qquad (17)$$

Assuming the peak $\mathcal{R}_m(s)$ has an area normalized to π, consistent with $\mathcal{R}_m(s) \to \pi\delta(s - m^2)$ in the narrow resonance limit, one finds that

$$\pi = \int_0^{s_0} ds \mathcal{R}_m(s) = \int_0^{\Gamma_{max}} d\Gamma' f(\Gamma') \qquad (18)$$

provided $s_0 > m^2 + m\Gamma_{max}$. Consequently, one can use (16) and (18) to demonstrate that replacing factors of $\pi\delta(s - m^2)$ in (1) with $\mathcal{R}_m(s)$ will not alter narrow-resonance approximation predictions (14) for F_0 and F_1:

$$[F_0^h(s_0)]_r \to \int_0^{s_0} ds (1/\pi) g_r [\mathcal{R}_m(s)]_r = g_r \qquad (19)$$

$$[F_1^h(s_0)]_r \to \int_0^{s_0} ds (1/\pi) s\ g_r [\mathcal{R}_m(s)]_r$$
$$= (1/\pi) \int_0^{\Gamma_{max}} d\Gamma' f(\Gamma') \qquad (20)$$
$$\times \int_0^{s_0} ds\ s\ g_r \mathcal{P}_{m_r}(s, \Gamma') = g_r m_r^2.$$

Thus we see that the first two finite-energy sum-rules F_0 and F_1 are impervious to resonance-width effects, provided the resonance in question is a symmetric peak that is entirely below the continuum threshold s_0. Consequently, we observe that these sum rules are particularly well-suited for an analysis of broad subcontinuum resonances.

3 FESR suppression of higher-dimensional condensates with one or more $\bar{q}q$-pairs

The operator-product expansion (OPE) for a dimension-2 two-current correlation function $\Pi(s)$ can be expressed at Euclidean momenta $Q^2 \equiv -s > 0$ in terms of QCD-vacuum condensates as follows:

$$\begin{aligned}
&\Pi(-Q^2) \\
&= C_p(Q^2) + C_{\bar{q}q}(Q^2) < m_q \bar{q}q > \\
&+ C_{G^2}(Q^2) < \alpha_s G^2 > + C_M(Q^2) < \bar{q}G \cdot \sigma q > \qquad (21) \\
&+ C_{G^3}(Q^2) < \alpha_s G^3 > + C_{(\bar{q}q)^2}(Q^2) < \alpha_s (\bar{q}q)^2 > + ...
\end{aligned}$$

To leading order in α_s, the OPE coefficients $C_n(Q^2)$ of an n-dimensional condensate $< O_n >$ are of the general form

$$C_n(Q^2) = \sum_j [A_j + B_j ln(Q^2/\mu^2)] m_q^j / Q^{j+n-2} \qquad (22)$$

To avoid mass singularities, the index j is restricted to zero and even positive integers if n is even, and to odd positive integers if n is odd. To leading order in α_s, contributions to (21) from condensates containing at least one fermion-antifermion pair necessarily correspond to diagrams with broken loops [Fig. 2], and for such diagrams $B_p = 0$; logarithms arising from integrations over closed-loop momenta do not occur. For example, to leading order in α_s, the $< m_q\bar{q}q >$ contribution [Fig. 2a] to the

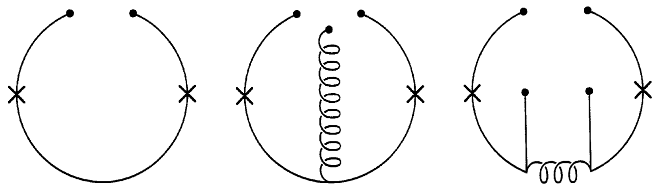

Fig. 2. a Leading $< \bar{q}q >$ contribution to current correlation functions, **b** typical leading $< \bar{q}G \cdot \sigma q >$ contribution to current correlation functions, **c** typical leading $< \alpha_s(\bar{q}q)^2 >$ contribution to current correlation functions

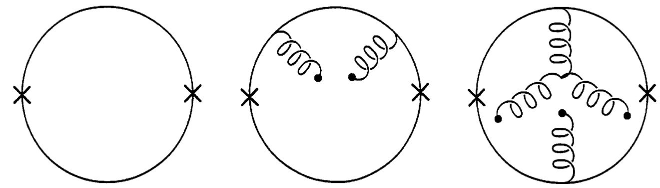

Fig. 3. a Leading purely-perturbative contribution to current correlation functions, **b** typical leading $< \alpha_s G^2 >$ contribution to current correlation functions, **c** typical leading $< G^3 >$ contribution to current correlation functions

longitudinal component $\Pi^L(s)$ of the axial-vector current correlation function,

$$\begin{aligned}(g_{\mu\nu} - p_\mu p_\nu/p^2)\Pi^T(p^2) + (p_\mu p_\nu/p^2)\Pi^L(p^2) \\ \equiv i \int d^4x \, e^{ip\cdot x} < 0|T j_{\mu 5}(x) j_{\nu 5}(0)|0 >, \end{aligned} \quad (23)$$

$[j_{\mu 5} \equiv \bar{u}\gamma_\mu \gamma_5 d]$ is given by [7]

$$\begin{aligned}C^L_{\bar{q}q}(Q^2) &= (2/m_q^2)[1 - (1 + 4m_q^2/Q^2)^{1/2}] \\ &= -4/Q^2 + 4m_q^2/Q^4 - 8m_q^4/Q^6 + \ldots \end{aligned} \quad (24)$$

If in (22) $B_j = 0$ for all j, the definition (13) implies that the FESR's F_0 and F_1 are (respectively) sensitive *only* to first and second order poles at $Q^2 = 0$:

$$\begin{aligned}[F_0^L(s_0)]_{\bar{q}q} &= (1/2\pi i) \int_{C(s_0)} ds \, C^L_{\bar{q}q}(-s) < m_q \bar{q}q > \\ &= -4 < m_q \bar{q}q > \end{aligned} \quad (25)$$

$$\begin{aligned}[F_1^L(s_0)]_{\bar{q}q} &= (1/2\pi i) \int_{C(s_0)} ds \, s \, C^L_{\bar{q}q}(-s) < m_q \bar{q}q > \\ &= -4m_q^2 < m_q \bar{q}q > \end{aligned} \quad (26)$$

Thus, if $C_n(Q^2)$, the OPE coefficient of a condensate $< O_n >$, is restricted to inverse powers of Q^2, then n must be less than or equal to 6 for that condensate to contribute to F_0 or F_1. If $n > 6$, then $n + j - 2 \geq 6$ and the leading OPE contribution to (22) is at least a third order pole at $Q^2 = 0$, which cannot contribute to F_0 or F_1.

For the particular case of the longitudinal component (L) of the axial-vector correlation function, which is coupled to pion-resonance states, there is an additional chiral symmetry constraint that $C_n^L(Q^2) \to 0$ as $m_q \to 0$, in which case $j \geq 1$ for all coefficients of condensates that fail to vanish in the chiral limit. As a consequence, one can show to leading order in α_s that the $n = 6$ condensate $< \alpha_s(\bar{q}q)^2 >$ cannot contribute to F_0 or F_1, as its leading contribution is necessarily a third-order pole at $Q^2 = 0$ [1, 8]:

$$C^L_{(\bar{q}q)^2}(Q^2) = -448\pi m_q^2 \alpha_s/27Q^6 + O(m_q^4/Q^8). \quad (27)$$

Similarly, F_0 and F_1 are found to be insensitive to the ($n = 5$) mixed condensate $< \bar{q}G \cdot \sigma q >$. The relevant contribution to the longitudinal component of the axial-vector correlator is also seen to involve only third-and-higher order poles at $Q^2 = 0$ [9]:

$$\begin{aligned}C^L_M(Q^2) &= -(1 - v)^3/2m_q^3 v \\ &= 4m_q^3/Q^6 - 20m_q^5/Q^8 + \ldots, \end{aligned} \quad (28)$$

$$v \equiv (1 + 4m_q^2/Q^2)^{1/2}. \quad (29)$$

Thus the leading contributions to F_0 and F_1 sum rules in this channel do not involve any condensates with quark-antiquark pairs except $< m_q \bar{q}q >$. The F_0 and F_1 sum rules in other channels can also involve the n = 5 mixed condensate $< \bar{q}G \cdot \sigma q >$ and the n = 6 condensate $< \alpha_s(\bar{q}q)^2 >$ [we are assuming vacuum-saturation], but no other condensates containing quark-antiquark pairs, as all other such condensates are of dimension greater than 6.

4 Purely gluonic contributions to F_0 and F_1

4.1 Purely perturbative gluon-loop contributions

For two-current correlation functions, the suppression of leading-order contributions from $n > 6$ condensates applies only to those operators whose leading contribution in α_s does not involve a closed perturbative loop. However, all condensates involving only gluons necessarily are generated from the closed-loop vacuum polarization diagram [Fig. 3], and such diagrams are characterized by nonzero coefficients B_j and the OPE expansion (22). The contribution of such logarithmic terms in (22) to the FESRs F_0 and F_1 can be obtained from the general relation [$Q^2 \equiv -s$; D is an integer]

$$\begin{aligned}&\int_{C(s_0)} ds \, ln(Q^2)/(Q^2)^D \\ &= -2i\pi(-1)^D s_0^{1-D}/(1 - D); \quad D \geq 2. \end{aligned} \quad (30)$$

However, a more precise evaluation of the contributions of closed-loop OPE coefficients necessarily involves the one-loop momentum integral

$$\begin{aligned}X(v) &\equiv (1/v^2) \int_0^1 dx \, ln[1 - sx(1 - x)/m_q^2 - i|\epsilon|] + 2/v^2, \\ v &\equiv (1 - 4m_q^2/s)^{1/2}. \end{aligned} \quad (31)$$

For Euclidean momenta ($s < 0$), $X(v)$ is given by the real function $X(v) = (1/v)ln[(1+v)/(v-1)]$. For Minkowskian momenta ($s > 0$), $X(v)$ develops an imaginary part above the quark-antiquark kinematic production threshold:

$$X(v) = (1/v)\left\{ln[(1+v)/(1-v)] - i\pi\right\},\ s > 4m_q^2. \quad (32)$$

The result (32) facilitates the sum-rule determination of closed loop contributions to $F_{0,1}$. For example, the one-loop purely perturbative contribution [Fig. 3a] to the longitudinal component of the axial-vector current correlator (29) is given by [9]

$$C_p^L[v] = (-3m_q^2/2\pi^2)[v^2 X(v) + \text{divergent constant}]. \quad (33)$$

The contribution of (33) to $F_{0,1}$ is easily obtained via a distortion of the contour $C(s_0)$ to that in Fig. 1b:

$$\begin{aligned}
&[F_0^L(s_0)]_p \\
&= (1/\pi)\int_0^{s_0} ds\ Im\left\{C_p^L[v]\right\} \\
&= (3m_q^2/2\pi^2)\int_{4m_q^2}^{s_0} ds(1-4m_q^2/s)^{1/2} \quad (34) \\
&= (3m_q^2/2\pi^2)\quad s_0(1-4m_q^2/s_0)^{1/2} + 2m^2 \\
&\quad \times ln\left|[1-(1-4m_q^2/s_0)^{1/2}]/[1+(1-4m_q^2/s_0)^{1/2}]\right|\Big\}
\end{aligned}$$

$$\begin{aligned}
&[F_1^L(s_0)]_p \\
&= (1/\pi)\int_0^{s_0} ds\ s\ Im\left\{C_p^L[v]\right\} \quad (35) \\
&= (3m_q^2/4\pi^2)(s_0^2 - 2m_q^2 s_0)(1-4m_q^2/s_0)^{1/2} + (3m_q^6/\pi^2) \\
&\quad \times ln\left|\Big[1-(1-4m_q^2/s_0)^{1/2}]/[1+(1-4m_q^2/s_0)^{1/2}\Big]\right|
\end{aligned}$$

We note that the results (34) and (35) are *exact* expressions obtained from the one-loop expression (33), which, to leading order in the quark mass m_q, yield the following:

$$\begin{aligned}
[F_0^L(s_0)]_p &= (3m_q^2 s_0/2\pi^2) + O(m_q^4), \\
[F_1^L(s_0)]_p &= (3m_q^2 s_0^2/4\pi^2) + O(m_q^4). \quad (36)
\end{aligned}$$

4.2 Two-gluon condensate contributions to $F_{0,1}$

The OPE coefficient $C_{G^2}(Q^2)$ is extracted from the OPE coefficient $E_{G^2}(Q^2)$ in the "normal-ordered basis" (i.e. the "heavy quark" coefficients listed in Appendix B of [9]) as follows:

$$\begin{aligned}
C_{G^2}(Q^2) = E_{G^2}(Q^2) &+ (1/12\pi)C_{\bar{q}q}(Q^2) \\
&- (m_q/2\pi)ln(m_q^2/\mu^2)C_M(Q^2). \quad (37)
\end{aligned}$$

The linear combination (37) represents the coefficient in the "minimally-subtracted basis", which is chosen so as to avoid mass-singularities [10]. For example, in the normal-ordered basis the coefficient of $< \alpha_s G^2 >$ for the longitudinal component of the axial-vector correlator is [9]

$$\begin{aligned}
E_{G^2}^L(Q^2) = &(-1/96\pi Q^2)[16m_q^4(3+9v^2)X(v)/(v^4Q^4)] \\
&+ [1/(48\pi v^4 Q^2)][9v^4 + 4v^2 + 3], \quad (38)
\end{aligned}$$

which generates an expansion

$$\begin{aligned}
E_{G^2}^L(Q^2) = &1/3\pi Q^2 - 5m_q^2/6\pi Q^4 + (m_q^4/\pi Q^6) \\
&\times[13/3 + 2ln(m_q^2/Q^2)] + O(m_q^6/Q^8). \quad (39)
\end{aligned}$$

The leading term on the right hand side of (39) does not vanish as $m_q \to 0$, despite the chiral invariance of $< \alpha_s G^2 >$. Moreover, the right hand side has the quark mass appear in the logarithm, which could (in principle) lead to a large logarithm after subtractions. The change of basis (37) eliminates both problems, as is evident from direct substitution of (39), (28) and (24) into (37):

$$\begin{aligned}
C_{G^2}(Q^2) = &m_q^2/2\pi Q^4 + (m_q^4/\pi Q^6) \\
&\times[11/3 - 2ln(Q^2/\mu^2)] + O(m_q^6/Q^8). \quad (40)
\end{aligned}$$

The result (40) is consistent with the general form (22), although the recipe (37) requires further modification if $O[m_q^6 ln(m_q^2/Q^2)/Q^8]$ terms are to be eliminated. It is worth noting that the change of basis (37) differs from an operator redefinition proposed on chiral symmetry grounds in [9] only by the presence of the final $C_M(Q^2)$ term, which has already been shown not to affect the contour integrals leading to F_0 and F_1. In Appendix A, the full contribution of $C_{G^2}^L(Q^2)$ to the $F_{0,1}$ sum rules for the longitudinal component of the axial-vector correlator is determined to all orders in m_q by careful consideration of the $C(s_0)$ contour. However, contributions to F_0 and F_1 from the $< \alpha_s G^2 >$ condensate can be evaluated to $O(m_q^4)$ from application of (30) and the Cauchy residue theorem to (40):

$$\begin{aligned}
[F_0^L(s_0)]_{<\alpha_s G^2>} &= (1/2\pi i) < \alpha_s G^2 > \int_{C(s_0)} ds C_{G^2}^L(-s) \\
&= (m_q^4/\pi s_0^2) < \alpha_s G^2 > \quad (41)
\end{aligned}$$

$$[F_1^L(s_0)]_{<\alpha_s G^2>} = [m_q^2/2\pi + 2m_q^4/\pi s_0] < \alpha_s G^2 > \quad (42)$$

4.3 Higher-dimensional gluon condensate contributions to $F_{0,1}$

As in (37), the OPE coefficient $C_{G^3}(Q^2)$ in the minimally-subtracted basis can be extracted from the coefficient $E_{G^3}(Q^2)$ in the normal-ordered "heavy quark" basis [9,11]:

$$\begin{aligned}
C_{G^3}(Q^2) = E_{G^3}(Q^2) &+ [1/(360\pi m_q^2)]C_{\bar{q}q}(Q^2) \\
&+ [1/(12\pi m_q)]C_M(Q^2). \quad (43)
\end{aligned}$$

This change of basis once again eliminates leading-order mass-singularities. To see this, we demonstrate application of (43) to FESR's by once again considering the relevant contributions to the longitudinal component of the axial-vector current correlation function. The OPE coefficient $E_{G^3}^L(Q^2)$ is given by [9]

$$E_{G^3}^L = \frac{-m_q^4}{24\pi Q^8 v^8} X(v)[7 + 23v^2 + 13v^4 + 5v^6]$$

$$+\frac{1}{2880\pi Q^4 v^8(1-v^2)} \times[105+65v^2-494v^4+266v^6+5v^8-75v^{10}] = \frac{1}{\pi Q^2}\left[\frac{1}{90m_q^2}-\frac{1}{90Q^2}-\frac{14m_q^2}{45Q^4}+O(m_q^4)\right]. \quad (44)$$

The leading term on the last line of (44) diverges as $m_q \to 0$, an explicit mass singularity. The next-to-leading term fails to vanish in the chiral limit. However, both of these terms, as well as the explicit $O(m_q^2)$ term in (44) cancel in (43) against corresponding terms from $C^L_{\bar{q}q}$ (24) and C^L_M (28). Consequently, the OPE coefficient $C^L_{G^3}$ is explicitly $O(m_q^4)$, and is therefore suppressed relative to $C^L_{G^2}$. The suppression of C_{G^3} relative to C_{G^2} in the operator product expansion appears to be a general property [11,12]. Factors of C_{G^3} for the scalar, vector, and axial-vector [transverse component] current correlation functions, as extracted via (43) from the "heavy quark" expressions in [9], are also seen to exhibit suppression by m_q^2 relative to corresponding factors of C_{G^2}.

By contrast, the dimension-8 gluon contributions to scalar, pseudoscalar and vector correlation functions (which in our conventions are defined to have dimensions of mass squared) are argued in [13] to be of the form $[A_0 + B_0 ln(Q^2/\mu^2)] < G^4 > /Q^6$, where A_0 and B_0 are numerical: suppression by m_q^2 does not seem to occur. For the *longitudinal* component of the axial-vector correlator, which picks up a factor of m_q^2/Q^2 relative to the pseudoscalar correlator, a dimension-8 contribution to F_1^L will then be proportional [via (30)] to $m_q^2 B_0 < G^4 > /s_0^2$. Such a contribution will be small compared to that of the dimension-4 condensate $< G^2 >$ [eq. (42)] provided $B_0 < G^4 >$ is small compared to $< G^2 > s_0^2$, suggesting, in the absence of m_q^2-suppression factors, that any further suppression of 2d-dimensional gluon condensates to FESR's is contingent upon the ratio $< G^{2d} > /s_0^{2d}$ being small. Such a small ratio can be anticipated via dimensional and factorization arguments [e.g. $< G^2 > < s_0^2$].

5 Direct single-instanton contributions to F_1^L

In the instanton liquid model, the direct single-instanton contribution to the R_0 Laplace sum rule (2) for the pseudoscalar (P) correlation function has been found to be [14]

$$R_0^P(\tau) \equiv (1/\pi)\int_0^\infty Im\left\{\Pi^P(s)]_{inst}\right\} e^{-s\tau} ds = \frac{3\rho^2}{8\pi^2\tau^3} e^{-\rho^2/2\tau}[K_0(\rho^2/2\tau)+K_1(\rho^2/2\tau)], \quad (45)$$

where ρ [= 1/(600 MeV)] is the instanton-size parameter. Since the pseuscalar correlator is related to the longitudinal component (L) of the axial-vector correlator by $\Pi^L(s) = 4m_q^2\Pi^P(s)/s$, we see that the instanton contribution $\mathcal{F}_1$ to the corresponding FESR F_1^L is [1]

$$\mathcal{F}_1(s_0) = (1/\pi)\int_0^{s_0} Im\left\{[\Pi^L(s)]_{inst}\right\} sds = (4m_q^2/\pi)\int_0^{s_0} Im\left\{[\Pi^P(s)]_{inst}\right\} ds, \quad (46)$$

which is related via Laplace transformation to the function (45) for $R_0^P(s)$ as follows:

$$\mathcal{F}_1'(t) = (4m_q^2/\pi) Im\left\{[\Pi^P(t)]_{inst}\right\}, \quad (47)$$

$$\mathcal{L}[\mathcal{F}_1'(t)] \equiv \int_0^\infty \mathcal{F}_1'(t)e^{-st}dt = 4m_q^2 R_0^P(s) = s\mathcal{L}[\mathcal{F}_1(t)] - \mathcal{F}_1(0). \quad (48)$$

We see from (46) that $\mathcal{F}_1(0) = 0$, and find that

$$\mathcal{F}_1(t) = \mathcal{L}^{-1}[4m_q^2 R_0^P(s)/s] \quad (49)$$
$$= \mathcal{L}^{-1}\left[\frac{3\rho^2 m_q^2}{2\pi^2 s^4} e^{-\rho^2/2s}[K_0(\rho^2/2s)+K_1(\rho^2/2s)]\right].$$

Our use of the variables s and t is to retain consistency with standard Laplace transform conventions; the variable t will ultimately be identified with the continuum threshold s_0, and the variable s corresponds to the Borel parameter τ in (45) [as defined in (2)]. The inverse transform of (49) may be obtained from the asymptotic expansions of K_0 and K_1 [15]:

$$K_0(z)+K_1(z) = (\pi/2z)^{1/2}e^{-z} \times[2+1/(4z)-3/(64z^2)+15/(512z^3)...] \quad (50)$$

$$\mathcal{F}_1(t) = \frac{3m_q^2\rho}{\pi^{3/2}}\left[\mathcal{L}^{-1}(s^{-7/2}e^{-\rho^2/s}) + \frac{1}{4\rho^2}\mathcal{L}^{-1}(s^{-5/2}e^{-\rho^2/s}) - \frac{3}{32\rho^4}\mathcal{L}^{-1}(s^{-3/2}e^{-\rho^2/s}) + \frac{15}{128\rho^6}\mathcal{L}^{-1}(s^{-1/2}e^{-\rho^2/s}) + ...\right] \quad (51)$$

Using (51) and replacing t with s_0, we find that

$$\mathcal{F}_1(s_0) = \frac{3m_q^2}{\pi^2\rho^4}G(2\rho s_0^{1/2}), \quad (52)$$

$$G(w) \equiv \left\{[-w^2/4+25/32+O(1/w^2)]sin(w) + [-7w/8+15/(64w)+O(1/w^3)]cos(w)\right\}. \quad (53)$$

[1] The direct single-instanton contributions to F_0^L and F_1^L are both $O(m_q^2)$. This implies that the instanton contribution to F_0^L is small in comparison to that of $< \bar{q}q >$ (25), which is why we focus here on F_1^L

The results (51-53) are not useful unless $w \to 2\rho\, s_0^{1/2} > 1$. Since $s_0^{1/2}$ is generally expected to be at least 1 GeV, the expansion in large w is appropriate and useful [$\rho^{-1} \cong 0.6$ GeV]. In the large s_0 limit, the leading perturbative contribution to F_1^L [eq. (36)] dominates the instanton contribution, which is at most linear in s_0 (52-3). However, for values of s_0 near 1 GeV^2, the instanton contribution is shown in the next section to be larger than the perturbative contribution, with phenomenological implications for the light quark mass.

6 Discussion: FESR's in the pseudoscalar channel and the light-quark mass

An old [16] and ongoing [17] controversy in sum rule applications concerns the failure of the field-theoretical content of the QCD sum rules to saturate the pseudoscalar channel. The essence of this problem is evident from a qualitative examination of the R_0 and R_1 Laplace sum rules for the longitudinal component of the axial-vector current correlation function, as defined in (2) and (4). For suitable values of the Borel parameter τ ($M \equiv \tau^{-1/2} >> m_\pi$), one finds that

$$\begin{aligned} R_0 &= f_\pi^2 m_\pi^2 + \sum_{M_i^2 < s_0} F_i^2 M_i^2 \exp(-M_i^2 \tau) \\ &= -4 < m_q \bar{q} q > + O(m_q^2), \end{aligned} \tag{54}$$

a result consistent with the current-algebra GMOR relationship $f_\pi^2 m_\pi^2 = -4 < m_q \bar{q} q >$ [18] as long as the subsequent subcontinuum resonances in the summation on the hadronic side of (54) are either sufficiently heavy ($M_i^2 >> 1/\tau >> m_\pi^2$), or their decay constants F_i^2 are sufficiently small ($F_i^2 << f_\pi^2 m_\pi^2 / M_i^2$). The leading field-theoretical contribution to the R_1 sum rule, however, is quadratic in the quark mass [1,8]:

$$\begin{aligned} R_1 &= f_\pi^2 m_\pi^4 + \sum_{M_i^2 < s_0} F_i^2 M_i^4 \exp(-M_i^2 \tau) \\ &= m_q^2 [-4 < m_q \bar{q} q > \\ &\quad + 3/(2\pi^2 \tau^2) + < \alpha_s G^2 > /2\pi \\ &\quad + 448 \pi \tau < \alpha_s (\bar{q} q)^2 > /27 + ...]. \end{aligned} \tag{55}$$

Naively, the field-theoretical content of (55) is of order m_q^2 times the field-theoretical content of (54), whereas the hadronic content of (55) is at least of order m_π^2 times the hadronic content of (54), suggesting that m_q and m_π are comparable. A thorough treatment of QCD contributions to (55) still yields substantially larger values of the light quark mass [8,19,20] than are anticipated from other phenomenology [5]. This mismatch in scale [i.e., $R_1^h / R_0^h \sim m_\pi^2$; $R_1^{QCD} / R_0^{QCD} \sim m_q^2$] superficially characterizes the FESR's F_0^L and F_1^L as well. However, these FESR's provide a much cleaner framework for extracting limits on m_q, enabling one to avoid the large-width modifications to the hadronic-resonance content of (55), as discussed in Sects. 1 and 2, as well as higher-dimensional condensate contributions (including that of $< \alpha_s (\bar{q} q)^2 >$) to the field-theoretical content of (55), as discussed in Sects. 3 and 4.

Direct single-instanton contributions (45) to the Laplace sum rule have been argued in a number of places [1,14,17] to be necessary for the saturation of the pseudoscalar channel. If we incorporate such contributions (52, 53) into the FESR F_1^L, in conjunction with the (width-independent) $k = 1$ hadronic contributions (14) as well as the leading [$O(m_q^2)$] field theoretical contributions (26, 36, 42), we find that

$$\begin{aligned} F_1^L &= f_\pi^2 m_\pi^4 + \sum_{M_i^2 < s_0} F_i^2 M_i^4 \\ &= m_q^2 [-4 < m_q \bar{q} q > + < \alpha_s G^2 > /2\pi + 3 s_0^2 / 4\pi^2 \\ &\quad + (3/\pi^2 \rho^4) G(2\rho s_0^{1/2}) + O(m_q)]. \end{aligned} \tag{56}$$

For each subcontinuum pion-excitation state, we define the resonance parameter $r_i \equiv (F_i^2 M_i^4)\ /(f_\pi^2 m_\pi^4)$. We can then rearrange (56) to obtain the following relationship for the light-quark mass:

$$m_q^2 = \frac{f_\pi^2 m_\pi^4 (1 + \sum r_i)}{A + [G(w) + w^4/64] B}, \tag{57}$$

$$\begin{aligned} &A \equiv -4 < m_q \bar{q} q > + < \alpha_s G^2 > /2\pi, \\ &B \equiv 3/(\pi^2 \rho^4). \end{aligned} \tag{58}$$

In (57), the summation is understood to be over only those resonance peaks below the continuum threshold ($M_i^2 < s_0$). The dependence on the continuum-threshold s_0 enters through the variable $w = 2\rho s_0^{1/2}$, and the function $G(w)$ is given by (53). Duality implies that the relationship (57) should retain approximate validity as s_0 increases to include the resonance peaks of additional pion-excitation states. In particular, one would expect the contribution from $\Pi(1300)$, the first pion-excitation (M = 1300 ± 100 MeV, Γ = 200 - 600 MeV [5]) to be fully subcontinuum if $s_0 > 4$ GeV^2. Possible additional contributions may accrue in full from $\Pi(1770)$ and X(1830) at even larger values of s_0.

Using standard parameter values [$< m_q \bar{q} q > = -f_\pi^2 m_\pi^2/4$, $< \alpha_s G^2 > = 0.045$ GeV^4, $\rho^{-1} = 600$ MeV], one can then estimate the following numerical lower bound on the quark mass from (57):

$$\begin{aligned} m_q &= \mu(w) \sqrt{1 + \sum\, r_i} \\ &> \mu(w) \sqrt{1 + r_1 \Theta(s_0 - 4GeV^2)}, \end{aligned} \tag{59}$$

$$\mu(w) = \frac{2.6 MeV}{\{0.0075 + 0.039 [G(w) + w^4/64]\}^{1/2}}, \tag{60}$$

where r_1 is the resonance parameter appropriate for the first pion-excitation state, as defined earlier. Although chiral Lagrangian arguments have been recently advanced suggesting that r_1 is substantially less than unity [4], sum-rule estimates for r_1 of order unity and larger [6] have received further support [21] from recent Laplace sum-rule fits.

Table 1. Behaviour of $\mu(w)$ with increasing s_0 in the presence (Column 3) and in the absence (Column 4) of direct single instanton contributions to the F_1^L finite energy sum rule

w	s_0 (GeV^2)	$\mu(w)(MeV)$ $G(w)$ given by (53)	$\mu(w)$ (MeV) $G(w)=0$
3.3	0.98	5.7	9.1
3.5	1.10	5.2	8.2
3.7	1.23	4.8	7.4
3.9	1.37	4.5	6.7
4.1	1.51	4.2	6.1
4.3	1.66	4.0	5.5
4.5	1.82	3.8	5.1
4.7	1.99	3.7	4.7
4.9	2.16	3.5	4.3
5.1	2.34	3.4	4.0
5.3	2.53	3.4	3.7
5.5	2.72	3.3	3.4
5.7	2.92	3.2	3.2
5.9	3.13	3.1	3.0
6.1	3.34	3.05	2.8
6.3	3.57	3.0	2.6
6.5	3.80	2.9	2.5
6.7	4.04	2.8	2.3

We reiterate that the FESR-based inequality (59) [and relation (57) from which it is derived] avoids any need for a narrow-resonance approximation, which would certainly be unphysical for dealing with broad subcontinuum pion-resonance states. The QCD-vacuum condensates that contribute are all lumped into the constant A (58); condensates such as $< \bar{q}G \cdot \sigma q >, < \alpha_s(\bar{q}q)^2 >$, and $< \alpha_s G^3 >$ do not generate any $O(m_q^2)$ contributions to F_1^L, as has already been discussed in Sects. 3 and 4. Even dimension-8 gluonic condensate contributions can be expected to be suppressed relative to those of $< \alpha_s G^2 >$ by the dimensional arguments presented at the end of Sect. 4.

In Table 1, we tabulate $\mu(w)$ for values of s_0 ranging from 1 GeV^2 to 4 GeV^2. We also tabulate the same function *in the absence of instanton contributions* [i.e. with $G(w) = 0$] in order to demonstrate the key role instantons play in obtaining a lighter and phenomenologically consistent quark mass over the entire range of s_0 considered. When the contribution of instantons is absent (Column 4 of Table 1), we find that $\mu(w)$ decreases from 9.1 MeV by a factor of four as s_0 increases from 1 GeV^2 to 4 GeV^2. This behaviour, if taken seriously, would not only suggest via (59) a rather large quark mass ($\sim$ 9 MeV), but also a very large aggregate contribution $\sum r_i \approx 15$ from subcontinuum resonance-peaks as s_0 increases to 4 GeV^2.

The instanton term $G(w)$ in the denominator of (60) greatly ameliorates these effects. When the instanton term is included (Column 3 of Table 1), we find that $\mu(w)$ decreases from 5.7 MeV by only a factor of two as s_0 goes from 1 GeV^2 to 4 GeV^2, suggesting via (59) a lighter ($\sim$ 6 MeV) quark mass in conjunction with a phenomenologically reasonable aggregate contribution $\sum r_i \approx 3$ from subcontinuum resonance-peaks as s_0 increases to 4 GeV^2. In view of the sparseness of such pion-resonance states [which suggests replacing $\sum r_i$ with r_1], it is noteworthy that this latter estimate is quite compatible with past [6] and present [21] sum rule estimates for r_1.

It is best to regard the results presented in this section as essentially qualitative. We have utilized only the one-loop-order purely-perturbative contribution to the correlation function Π^L – higher-order terms can be expected to alter the coefficient of the w^4-dependence in the denominator of (60). Inclusion of the renormalization group (RG) dependence of the running quark mass also lowers somewhat the size of the aggregate resonance contribution $\sum r_i$ as $s_0 \to 4$ GeV^2. Assuming $\Lambda_{QCD} = 0.2$ GeV, we find near-constancy of the RG-invariant quark mass $\hat{m}$ $\{m_q \to m_q(s_0) = \hat{m}/[ln(\sqrt{s_0}/\Lambda_{QCD})]^{4/9}\}$ over the 1 GeV$^2 \leq s_0 \leq 4$ GeV^2 range of Table 1 provided $\sum r_i \to 2$ if instantons are included, with $\sum r_i \to 10$ if instantons are not included. The key point here, however, is that the function $G(w)$ arising from instantons is oscillatory (53), going from positive to negative value as s_0 increases from 1 GeV^2 to 4 GeV^2. Moreover, $G(w)$ is not only positive, but is also larger over the range 1 GeV$^2 \leq s_0 \leq 1.6$ GeV^2 than the factor $w^4/64$ arising from perturbation theory, thereby lowering and stabilizing the quark mass (59) in a region for which there is at most only a partial contribution from the lowest subcontinuum resonance.

Acknowledgements. ASD, VE and TGS are grateful for support from the Natural Sciences and Engineering Research Council of Canada. VE is also grateful for several useful discussions with V. A. Miransky. AHF and YX wish to acknowledge research funding from the Faculty of Graduate Studies of the University of Western Ontario.

Appendix A: Evaluation of the gluon condensate contribution to F_0^L, F_1^L

The "heavy-quark" (h.q.) gluon condensate contribution to Π_L, as defined in (23), is obtained from Appendix B.3 of [9] as the sum of coefficients $[C_{1G^2}]_{h.q.}$ and $[C_{2G^2}]_{h.q.}$ for the axial-vector current correlation function:

$$\left[\Pi_L(p^2)\right]_{G^2} = (C_{1G^2} + C_{2G^2})_{h.q.} < G^2 >, \tag{A.1}$$

$$[C_{1G^2} + C_{2G^2}]_{h.q.} \equiv \alpha E_{G^2} = \alpha E_{pole} + \alpha C_x X(v), \tag{A.2}$$

$$\alpha E_{pole} = -\frac{\alpha}{96\pi}\left[\frac{18}{s} + \frac{14}{s-4m^2} + \frac{24m^2}{(s-4m^2)^2}\right], \tag{A.3}$$

$$\alpha C_x = \frac{\alpha}{2\pi} m^4 \left[\frac{1}{s^3 v^4} + \frac{3}{s^3 v^2}\right]. \tag{A.4}$$

We have extracted a factor of α so that E_{G^2} as defined in (A.2) is consistent with E_{G^2} as defined in Sect. 3.

The gluon condensate contribution to the finite energy sum rules

$$F_0^L = \frac{1}{2\pi i}\int_{C(s_0)} \Pi_L(s)\, ds,$$
$$F_1^L = \frac{1}{2\pi i}\int_{C(s_0)} \Pi_L(s)\, s\, ds, \tag{A.5}$$

can be obtained via (37) from direct evaluation of the integrals

$$G_0 \equiv \int_{C(s_0)} E_{G^2}\, ds, \quad G_1 \equiv \int_{C(s_0)} E_{G^2}\, s\, ds, \qquad (A.6)$$

with the contour $C(s_0)$ distorted as in Fig. 4 to encompass any pole singularities of E_{G^2} at $s = 0$ or $4m^2$ as well as the branch singularity for $s > 4m^2$. Using (32), one finds that

$$G_0 = -2i\pi \int_{4m^2+\epsilon}^{s_0} \frac{C_x}{v} ds + \int_{C_0} E_{pole} ds + \int_{C_{4m^2}} E_{pole} ds + \int_{C_0} C_x X(v) ds + \int_{C_{4m^2}} C_x X(v) ds, \qquad (A.7)$$

where the contours C_0 and C_{4m^2} are clockwise circles of radius ϵ about $s = 0$ and $s = 4m^2$, respectively (Fig. 4). We see from (A.3) that

$$\int_{C_0} E_{pole} ds = \frac{3i}{8}, \quad \int_{C_{4m^2}} E_{pole} ds = \frac{7i}{24}. \qquad (A.8)$$

Using the expression for C_x in (A.4), we find that

$$-2i\pi \int_{4m^2+\epsilon}^{s_0} \frac{C_x}{v} ds = -\frac{i}{8}\left[-\frac{1}{3v_0^3} - \frac{2}{v_0} - 3v_0\right] - \frac{im^3}{3\epsilon^{3/2}} - \frac{5im}{8\epsilon^{1/2}} + O(\epsilon^{1/2}) \qquad (A.9)$$

where $v_0 \equiv \sqrt{1 - 4m^2/s_0}$. The integral around the origin is straightforward to obtain from (A.4). The integrand

$$C_x X(v) = \frac{1}{16\pi}\left[2 + I(s)\right] \qquad (A.10)$$
$$\times \left[\frac{-3/2}{s - 4m^2} + \frac{6m^2}{(s-4m^2)^2} + \frac{8m^4}{(s-4m^2)^3} + \frac{3}{2s}\right],$$

$$I(s) = \int_0^1 dx\, ln\left[1 - sx(1-x)/m^2 - i|\epsilon|\right], \qquad (A.11)$$

has a simple pole at $s = 0$ because $I(0) = 0$:

$$\int_{C_0} C_x X(v) ds = -\frac{3i}{8}. \qquad (A.12)$$

Note that (A.12) *exactly cancels* the C_0 pole contribution (A.8), indicating that the origin can be excised from the contour of Fig. 4.

This cancellation is not peculiar to the channel we are in. We have verified (Appendix B) that an identical cancellation occurs in the scalar, vector, and transverse-axial channels between the contributions of explicit $s = 0$ poles in E_{G^2} [as in (A.8)] and the integrals of $C_x X(v)$ portions of E_{G^2} around C_0 [as in (A.12)]. Thus the quantum-field-theoretical singularities in G_0 and G_1 all occur for $s \geq 4m^2$ on the real s-axis for *all* of the above-mentioned channels.

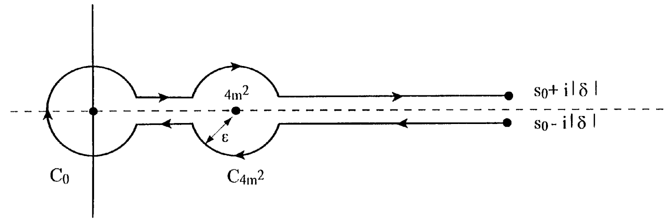

Fig. 4. Distortion of the $C(s_0)$ contour [Fig. 1a] for $< \alpha_s G^2 >$ contributions to $F_{0,1}$ sum rules

The divergence as $\epsilon \to 0$ in (A.9) is cancelled exactly by the integration of $C_x X(v)$, as given in (A.10), over the contour C_{4m^2} around $s = 4m^2$, a cancellation which also occurs in the other three channels mentioned above. This cancellation is most easily seen by continuing the expression (A.11) to complex values of s in the vicinity of $s = 4m^2$:

$$I(s) + 2 = 2\left[(4m^2 - s)/s\right]^{1/2} \tan^{-1}\left(\frac{s}{4m^2 - s}\right)^{1/2} = \pi\left(\frac{4m^2 - s}{s}\right)^{1/2} - 2\left(\frac{4m^2 - s}{s}\right) + \frac{2}{3}\left(\frac{4m^2 - s}{s}\right)^2 + ... \qquad (A.13)$$

Upon substitution of (A.13) into (A.10) one finds that

$$\int_{C_{4m^2}} C_x X(v) ds = -\frac{7i}{24} + \frac{1}{16}\left[\frac{3}{2}\int_{C_{4m^2}} s^{-1/2}(4m^2 - s)^{-1/2} ds + 6m^2 \int_{C_{4m^2}} s^{-1/2}(4m^2 - s)^{-3/2} ds - 8m^4 \int_{C_{4m^2}} s^{-1/2}(4m^2 - s)^{-5/2} ds + \frac{3}{2}\int_{C_{4m^2}} s^{-3/2}(4m^2 - s)^{1/2} ds\right]. \qquad (A.14)$$

The factor $-7i/24$ is just $-2\pi i$ times the aggregate residue at $s = 4m^2$ obtained from multiplication of (A.13)'s integer powers of $(4m^2 - s)$ into (A.10). This pole contribution *explicitly cancels* the C_{4m^2} pole contribution (A.8). The remaining integrals in (A.14) result from multiplying the leading $\pi[(4m^2 - s)/s]^{1/2}$ term of (A.13) into (A.10). They are evaluated by noting that $s = 4m^2 + \epsilon e^{i\theta}$ on the contour C_{4m^2}, with a clockwise rotation of θ from 2π to 0. When $s > 4m^2$, the correct (negative) sign of the imaginary part $[2iIm\{I(s)\} \equiv I(s + i|\delta|) - I(s - i|\delta|)]$ is obtained by requiring that $(4m^2 - s)^{1/2} = -i\epsilon^{1/2}e^{i\theta/2}$:

$$\int_{C_{4m^2}} (4m^2 - s)^{-1/2} s^{-1/2} ds = O(\epsilon^{1/2}), \qquad (A.15)$$

$$\int_{C_{4m^2}} (4m^2 - s)^{-3/2} s^{-1/2} ds = \frac{2i}{m\epsilon^{1/2}} + O(\epsilon^{1/2}), \quad (A.16)$$

$$\int_{C_{4m^2}} (4m^2 - s)^{-5/2} s^{-1/2} ds = -\frac{2i}{3m\epsilon^{3/2}} + \frac{i}{4m^3\epsilon^{1/2}} + O(\epsilon^{1/2}), \quad (A.17)$$

$$\int_{C_{4m^2}} (4m^2 - s)^{1/2} s^{-3/2} ds = O(\epsilon^{3/2}). \quad (A.18)$$

Substituting (A.15 - A.18) into (A.14) we find that

$$\int_{C_{4m^2}} C_x X(v) ds = -\frac{7i}{24} + \frac{5im}{8\epsilon^{1/2}} + \frac{im^3}{3\epsilon^{3/2}} + O(\epsilon^{1/2}), \quad (A.19)$$

explicitly cancelling the divergencies in (A.9). Since all the $s = 0$ and $s = 4m^2$ pole terms contributing to G_0 have also been shown to cancel, we find that G_0 is equal to the upper-bound contribution of the first integral on the right-hand side of (A.7):

$$G_0 = -2i\pi \int^{s_0} \frac{C_x}{v} ds = \frac{i}{8} \left[\frac{1}{3v_0^3} + \frac{2}{v_0} + 3v_0\right]; \quad v_0 \equiv \sqrt{1 - 4m^2/s_0}. \quad (A.20)$$

To obtain the full contribution of $< \alpha_s G^2 >$ to the F_0 sum rule, we substitute (37) from the text into (41), utilizing the results (A.6) and (A.20) in conjunction with (24) and (28) from the text:

$$\left[F_0^L(s_0)\right]_{<\alpha_s G^2>} = < \alpha_s G^2 > \left\{\frac{1}{16\pi}\left[\frac{1}{3v_0^3} + \frac{2}{v_0} + 3v_0\right] - \frac{1}{3\pi}\right\} = < \alpha_s G^2 > \left[\frac{m^4}{\pi s_0^2} + \frac{14m^6}{3\pi s_0^3} ...\right]. \quad (A.21)$$

To find the gluon condensate contribution to F_1^L, consider first the integral G_1 (A.6), which can be evaluated via the following integrals arising from the distortion of $C(s_0)$ indicated in Fig. 4:

$$G_1 = -2i\pi \int_{4m^2+\epsilon}^{s_0} \frac{C_x}{v} s ds + \int_{C_0} E_{pole} s\, ds + \int_{C_{4m^2}} E_{pole} s\, ds + \int_{C_0} C_x X(v) s\, ds + \int_{C_{4m^2}} C_x X(v) s\, ds. \quad (A.22)$$

One sees from (A.3) that

$$\int_{C_0} E_{pole} s\, ds = 0\,, \quad \int_{C_{4m^2}} E_{pole} s\, ds = \frac{5im^2}{3}. \quad (A.23)$$

Using the expression for C_x in (A.4), we find that

$$-2i\pi \int_{4m^2+\epsilon}^{s_0} \frac{C_x}{v} s ds = im^2 \left[\frac{1}{6v_0^3} + \frac{3}{2v_0}\right] - \frac{4im^5}{3\epsilon^{3/2}} - \frac{7im^3}{2\epsilon^{1/2}}. \quad (A.24)$$

Using (A.10), we find that $C_x X(v) s$ has no poles at $s = 0$ [note that $2+I(0) = 2$], in which case $\int_{C_0} C_x X(v) s\, ds = 0$. Once again, we note that the origin can be excised entirely from the contour of Fig. 4. The divergence in (A.24) as $\epsilon \to 0$ is exactly cancelled by integration of $C_x X(v) s$ around the contour C_{4m^2}. From (A.10) we find that

$$C_x X(v) s = \frac{2}{\pi}[2 + I(s)] \times \left[\frac{m^4}{(s - 4m^2)^2} + \frac{m^6}{(s - 4m^2)^3}\right]. \quad (A.25)$$

If we substitute (A.13) into (A.25) and integrate around C_{4m^2}, we easily separate a pure-pole contribution from an ϵ-dependent contribution involving half-integral powers of $(4m^2 - s)$:

$$\int_{C_{4m^2}} C_x X(v) s\, ds = -\frac{5im^2}{3} + \frac{7im^3}{2\epsilon^{1/2}} + \frac{4im^5}{3\epsilon^{3/2}} + O(\epsilon^{1/2}). \quad (A.26)$$

(A.26) is obtained through use of (A.16) and (A.17). Not only are the ϵ-dependent terms in (A.24) cancelled by (A.26), but the C_{4m^2} pole contribution (A.23) also cancels against the pole term in (A.26). Thus we find that G_1 is also equal to the upper-bound contribution of the first integral on the right-hand side of (A.22):

$$G_1 = -2i\pi \int^{s_0} \frac{C_x s}{v} ds = im^2 \left[\frac{1}{6v_0^3} + \frac{3}{2v_0}\right]. \quad (A.27)$$

We substitute (37) of the text into the integral in (42), utilizing the results (A.27) in conjunction with (24) and (28) from the text:

$$\left[F_1^L(s_0)\right]_{<\alpha_s G^2>} = \frac{m^2}{2\pi} < \alpha_s G^2 > \left\{\left[\frac{1}{6v_0^3} + \frac{3}{2v_0}\right] - \frac{2}{3}\right\} \quad (A.28)$$
$$= \frac{m^2}{2\pi} < \alpha_s G^2 > \left\{1 + \frac{4m^2}{s_0} + \frac{14m^4}{s_0^2} + \frac{160m^6}{3s_0^3} ...\right\}.$$

Appendix B: Gluon condensate contributions to $F_{0,1}$ in other channels

Scalar Channel

From Appendix B.1 of [9], we have

$$[C_{G^2}]_{h.q.} \equiv \alpha E_{G^2} = \alpha \left(E_{pole} + C_x X(v)\right), \quad (B.1)$$

$$E_{pole} = \frac{(3 - v^2)}{16\pi s v^2}, \quad C_x = -\frac{(1 - v^2)(3 + v^2)}{32\pi s v^2}. \quad (B.2)$$

We find that

$$\int_{C_0} E_{pole} ds = \frac{i}{8}, \quad \int_{C_{4m^2}} E_{pole} ds = -\frac{3i}{8}, \quad (B.3)$$

$$-2i\pi \int_{4m^2+\epsilon}^{s_0} \frac{C_x}{v} ds = \frac{i}{8}\left[-\frac{3}{v_0} + v_0 + \frac{6m}{\sqrt{\epsilon}}\right], \qquad \text{(B.4)}$$

$$\int_{C_0} C_x X(v) ds = -\frac{i}{8},$$

$$\int_{C_{4m^2}} C_x X(v) ds = \frac{3i}{8}\left(1 - \frac{2m}{\sqrt{\epsilon}}\right). \qquad \text{(B.5)}$$

Summing these integrals, we obtain

$$G_0 = \int_{C(s_0)} E_{G^2} ds = \frac{i}{8}\left[-\frac{3}{v_0} + v_0\right]. \qquad \text{(B.6)}$$

We also find from Appendix B.1 of [9] that

$$\int_{C(s_0)} C_{\bar{q}q} ds = 6i\pi, \quad \int_{C(s_0)} C_M ds = 0, \qquad \text{(B.7)}$$

which implies via (37) that

$$[F_0(s_0)]_{<\alpha_s G^2>} = \frac{1}{16\pi}\left[-\frac{3}{v_0} + v_0 + 4\right] < \alpha_s G^2 > . \qquad \text{(B.8)}$$

Unlike the case of F_0, the FESR F_1 requires the use of (37) to eliminate a logarithmic mass singularity in G_1, obtained by summing the following five integrals:

$$\int_{C_0} E_{pole} s\, ds = 0, \quad \int_{C_{4m^2}} E_{pole} s\, ds = -\frac{3im^2}{2} \qquad \text{(B.9)}$$

$$-2i\pi \int_{4m^2+\epsilon}^{s_0} \frac{C_x}{v} s\, ds = \frac{im^2}{2}\left[-\frac{3}{v_0} + 2\ell n(1 - v_0^2) + \frac{6m}{\sqrt{\epsilon}}\right], \qquad \text{(B.10)}$$

$$\int_{C_0} C_x X(v) s\, ds = 0,$$

$$\int_{C_{4m^2}} C_x X(v) s ds = \frac{3im^2}{2} - \frac{3m^3}{\sqrt{\epsilon}}. \qquad \text{(B.11)}$$

We then find that

$$G_1 = \int_{C(s_0)} E_{G^2} s\, ds = \frac{im^2}{2}\left[-\frac{3}{v_0} + 2\ell n\left(\frac{4m^2}{s_0}\right)\right], \qquad \text{(B.12)}$$

which is not analytic in m at $m = 0$. However the results

$$\int_{C(s_0)} C_{\bar{q}q} s\, ds = 4im^2\pi,$$

$$\int_{C(s_0)} C_M s\, ds = -2im\pi, \qquad \text{(B.13)}$$

used in conjunction with (37) in the contour integral appearing in (42) eliminates the quark-mass from the logarithm:

$$[F_1(s_0)]_{<\alpha_s G^2>} = \frac{m^2}{2\pi}\left[-\frac{3}{2v_0} + \frac{1}{3} - \ell n\left(\frac{s_0}{4\mu^2}\right)\right] < \alpha_s G^2 > . \qquad \text{(B.14)}$$

Transverse Axial Channel

From Appendix B.3 of [9], we have

$$[C_{1G^2}]_{h.q.} \equiv \alpha E_{G^2} = \alpha\left(E_{pole} + C_x X(v)\right), \qquad \text{(B.15)}$$

$$E_{pole} = -\frac{(1+v^2)}{8\pi s v^2}, \quad C_x = \frac{(1-v^2)^2}{16\pi s v^2}. \qquad \text{(B.16)}$$

We then find that

$$\int_{C_0} E_{pole} ds = \frac{i}{4}, \quad \int_{C_{4m^2}} E_{pole} ds = \frac{i}{4}, \qquad \text{(B.17)}$$

$$-2i\pi \int_{4m^2+\epsilon}^{s_0} \frac{C_x}{v} ds = \frac{i}{4}\left[\frac{1}{v_0} - v_0 - \frac{2m}{\sqrt{\epsilon}}\right], \qquad \text{(B.18)}$$

$$\int_{C_0} C_x X(v) ds = -\frac{i}{4},$$

$$\int_{C_{4m^2}} C_x X(v) ds = -\frac{i}{4} + \frac{im}{2\sqrt{\epsilon}}. \qquad \text{(B.19)}$$

As before, pole contributions from C_0 and C_{4m^2} are cancelled by (B.19), and the contour-radius singularity as $\epsilon \to 0$ cancels between (B.18) and (B.19):

$$G_0 = \int_{C(s_0)} E_{G^2} ds = \frac{i}{4}\left[\frac{1}{v_0} - v_0\right]. \qquad \text{(B.20)}$$

Since in this channel, one finds that [9]

$$\int_{C(s_0)} C_{\bar{q}q} ds = -4\pi i, \quad \int_{C(s_0)} C_M ds = 0, \qquad \text{(B.21)}$$

we find via (37) that

$$[F_0(s_0)]_{<\alpha_s G^2>} = \frac{1}{8\pi}\left[\frac{1}{v_0} - v_0 - \frac{4}{3}\right] < \alpha_s G^2 > . \qquad \text{(B.22)}$$

Corresponding results for F_1 are listed below:

$$\int_{C_0} E_{pole} s\, ds = 0, \quad \int_{C_{4m^2}} E_{pole} s\, ds = im^2, \qquad \text{(B.23)}$$

$$-2i\pi \int_{4m^2+\epsilon}^{s_0} \frac{C_x}{v} s\, ds = im^2\left(\frac{1}{v_0} - \frac{2m}{\sqrt{\epsilon}}\right), \qquad \text{(B.24)}$$

$$\int_{C_0} C_x X(v) s\, ds = 0,$$

$$\int_{C_{4m^2}} C_x X(v) s\, ds = -im^2 + \frac{2im^3}{\sqrt{\epsilon}}, \qquad \text{(B.25)}$$

$$\int_{C(s_0)} C_{\bar{q}q} s\, ds = -\frac{8i\pi m^2}{3}, \quad \int_{C(s_0)} C_M s\, ds = 0, \qquad \text{(B.26)}$$

$$[F_1(s_0)]_{<\alpha_s G^2>} = \left(\frac{m^2}{2\pi v_0} - \frac{m^2}{9\pi}\right) < \alpha_s G^2 > . \qquad \text{(B.27)}$$

Vector Channel

From (II.19) of [9], we find that

$$E_{pole} = -\frac{(3-2v^2+3v^4)}{48\pi s v^4},$$
$$C_x = \frac{(1-v^2)^2(1+v^2)}{32\pi s v^4}. \quad \text{(B.28)}$$

We then find that:

$$\int_{C_0} E_{pole} ds = \frac{i}{8}, \quad \int_{C_{4m^2}} E_{pole} ds = \frac{i}{24}, \quad \text{(B.29)}$$

$$-2i\pi \int_{4m^2+\epsilon}^{s_0} \frac{C_x}{v} ds = \frac{i}{8}\left[v_0 + \frac{1}{3v_0^3} - \frac{8m^3}{3\epsilon^{3/2}} - \frac{m}{\epsilon^{1/2}}\right]. \quad \text{(B.30)}$$

$$\int_{C_0} C_x X(v) ds = -\frac{i}{8},$$
$$\int_{C_{4m^2}} C_x X(v) ds = -\frac{i}{24} + \frac{im^3}{3\epsilon^{3/2}} + \frac{im}{8\epsilon^{1/2}}, \quad \text{(B.31)}$$

$$\int_{C(s_0)} C_{\bar{q}q} ds = -4i\pi, \quad \int_{C(s_0)} C_M ds = 0, \quad \text{(B.32)}$$

$$[F_0(s_0)]_{<\alpha_s G^2>} = \frac{1}{16\pi}\left(v_0 + \frac{1}{3v_0^3} - \frac{8}{3}\right) < \alpha_s G^2 > . \quad \text{(B.33)}$$

Corresponding results for F_1 are listed below:

$$\int_{C_0} E_{pole} s\, ds = 0. \quad \int_{C_{4m^2}} E_{pole} s\, ds = \frac{2im^2}{3}, \quad \text{(B.34)}$$

$$-2i\pi \int_{4m^2+\epsilon}^{s_0} \frac{C_x}{v} s\, ds = \frac{im^2}{2}\left[\frac{1}{v_0} + \frac{1}{3v_0^3} - \frac{8m^3}{3\epsilon^{3/2}} - \frac{3m}{\epsilon^{1/2}}\right], \quad \text{(B.35)}$$

$$\int_{C_0} C_x X(v) s\, ds = 0,$$
$$\int_{C_{4m^2}} C_x X(v) s\, ds = -\frac{2im^2}{3} + \frac{4im^5}{3\epsilon^{3/2}} + \frac{3im^3}{2\epsilon^{1/2}}, \quad \text{(B.36)}$$

$$\int_{C(s_0)} C_{\bar{q}q} s\, ds = -\frac{16}{3} i\pi m^2, \quad \int_{C(s_0)} C_M s\, ds = 0, \quad \text{(B.37)}$$

$$[F_1(s_0)]_{<\alpha_s G^2>} = \frac{m^2}{4\pi}\left[\frac{1}{v_0} + \frac{1}{3v_0^3} - \frac{8}{9}\right] < \alpha_s G^2 > . \quad \text{(B.38)}$$

References

1. M.A. Shifman, A.I. Vainshtein, V.I. Zakharov: Nucl. Phys. B **147** (1979) 385 and 448
2. P. Pascual, R. Tarrach: QCD: Renormalization for the practitioner, Berlin, Heidelberg, New York: Springer 1984, pp. 168–191
3. A.A. Andrianov, V.A. Andrianov: Nucl. Phys. B (Proc Supp) **39** B,C (1995) 257
4. M.K. Volkov, C. Weiss: Phys. Rev. D **56** (1997) 221
5. R.M. Barnett et al, [Particle Data Group]: Phys. Rev. D **54** (1996) 1
6. W. Hubschmid, S. Mallik: Nucl. Phys. B **193** (1981) 368
7. V. Elias, J.L. Murison, M.D. Scadron, T.G. Steele: Z. Phys. C **60** (1993) 235
8. C. Becchi, S. Narison, E. de Rafael, F.J. Yndurain: Z. Phys. C **8** (1981) 335
9. E. Bagán, J.I. LaTorre, P. Pascual: Z. Phys. C **32** (1986) 43
10. M. Jamin, M. Münz: Z. Phys. C **60** (1993) 569
11. S.C. Generalis, D.J. Broadhurst: Phys. Lett. B **139** (1984) 85
12. M.S. Dubovikov, A.V. Smilga: Nucl. Phys. B **185** (1981) 109; W. Hubschmid, S. Mallik: Nucl. Phys. B **207** (1982) 29; D.J. Broadhurst, S.C. Generalis: Phys. Lett. B **142** (1984) 75
13. D.J. Broadhurst, S.C. Generalis: Phys. Lett. B **165** (1985) 175
14. E.V. Shuryak: Nucl. Phys. B **214** (1983) 237; A.E. Dorokhov, S.V. Esaibegyan, N.I. Kochelev, N.G. Stefanis: J. Phys. G **23** (1997) 643
15. I.S. Gradshteyn, I. M Ryzhik: Table of integrals, series and products. Orlando: Academic Press 1980, p. 963 [eq. 8.451. 6]
16. V.A. Novikov, M.A. Shifman, A.I. Vainshtein, V.I. Zakharov: Nucl. Phys. B **191** (1981) 301
17. E.V. Shuryak: Rev. Mod. Phys. **65** (1993)1; K. Yamawaki, V.I. Zakharov, hep-ph/9406373 (unpublished)
18. M. Gell-Mann, R.J. Oakes, B. Renner: Phys. Rev. **175** (1968) 2195
19. S. Narison, E. de Rafael: Phys. Lett. B **103** (1981) 57
20. C.A. Dominguez, E. de Rafael: Ann. Phys. **174** (1987) 372; S. Narison: Phys. Lett. B **216** (1989) 191
21. V. Elias, A.H. Fariborz, M.A. Samuel, F. Shi, T.G. Steele: hep/9706472 (to appear in Phys. Lett. B); T.G. Steele, J.C. Breckenridge, M. Benmerrouche, V. Elias, A.H. Fariborz: hep-ph/9708454 (to appear in Nucl. Phys. A)